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Sharp versus smooth synchronization transition of locally coupled oscillators

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We provide a general condition for the occurrence of a sudden transition to synchronization in an array of oscillators mutually coupled via the nearest neighbors. At the onset of synchronization a specific constraint must be fulfilled: precisely, the response time of a single system to signals from the adjacent sites must be smaller than the refractory period. We verify this criterion in some models for neuronal dynamics, namely, in excitable systems driven by noise as well as in chaotic oscillators.

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Under particular conditions coupled systems exhibit coherent global behavior by synchronizing their dynamics. Coherent oscillations have been discussed in many areas. Winfree and Kuramoto [1] have shown that synchronization in globally coupled oscillators is reached suddenly, implying the existence of a phase transition. This discovery motivated many physical models to describe the synchronization of neuronal activity as the coordinating mechanism for feature binding [2], whereby spatially segregated processing areas are bound together to provide a coherent percept [3]. Due to the rapid onset of feature binding [4], neural synchronization shares aspects specific of a bifurcation or a phase transition, when driven by noise.

A sudden transition to synchronization, often referred to a phase transition, was found in many types of globally coupled systems, from phase rotators [1] to models of bursting neurons [5] and chaotic oscillators [6]. However, the global (all-to-all) coupling is a conceptual abstraction which has no anatomical correspondence in the brain wiring. Recently, a few cases of a sudden transition to synchronization in a locally coupled system have been reported. These are a large network of locally pulse coupled oscillators [7], locally coupled chaotic maps [8], and oscillators exhibiting homoclinic chaos [9]. At variance with global coupling, a locally coupled system does not generally display a sudden transition to synchronization but with a very few exceptions.

The aim of our study is to provide a general criterion for the occurrence of a bifurcation during a sudden transition to synchronization in locally coupled oscillators. We prove that in such a case the bifurcation occurs when a specific constraint is fulfilled, namely, as soon as the generation time becomes smaller than the refractory period. We verify this criterion in nonautonomous excitable systems driven by noise as well as in autonomous chaotic oscillators. The importance of a refractory period has been shown in some experimental studies provided recently [10]. The macaque visual area V4 displays two types of active neurons during attention. They have different refractory periods (<200 ms and >200 ms) and are called narrow and broad spiking neurons, respectively.

Let us first consider a one-dimensional array of coupled Adler systems [11],

$$\dot{x}_i = \mu - \cos x_i + D_i \xi_i + \epsilon \sum_{j \in n} \sin(x_{i+j} - x_i), \qquad (1)$$

where $n=\{-1,1\}$ for the bidirectional coupling. The variable x_i is a dimensionless angle (modulo 2π) and μ is a control parameter. For $|\mu|<1$, there are two fixed points at $x_\pm = \pm \arccos \mu$, one being a stable focus (x_-) and the other an unstable saddle point x_+. If $|\mu|>1$, there are no fixed points, and the flow consists of an oscillation of the variable x. This limit cycle develops through a saddle node on an invariant circle (Andronov) bifurcation at $\mu = \pm 1$, where the two fixed points collide and annihilate. For $|\mu|<1$, the system displays excitable behavior. ξ_i is a Gaussian white noise source of amplitude D_i. We consider the case where only the first site in the array is excited by an external forcing ($D_i=0.5$ for $i=0$), meanwhile the other units are excited by mutual interactions. The amplitude D_i is chosen in a such way that it triggers irregular spiking at the first site. It is worth noting that such an excitation can propagate in an array as soon as all Adler units are set with system parameters in an excitable dynamical regime. We use open boundary conditions.

In Fig. 1 we show the space-time positions of the spikes for different values of the coupling strength. For $\epsilon=0.75$ [Fig. 1(a)] an array slightly synchronizes, but rather random clusters are formed. For $\epsilon=1.05$ [Fig. 1(b)] we observe the emergence of synchronization, that means that the perturbation from the first site propagates through the whole array, reaching after some time the last site. By "synchronized" we

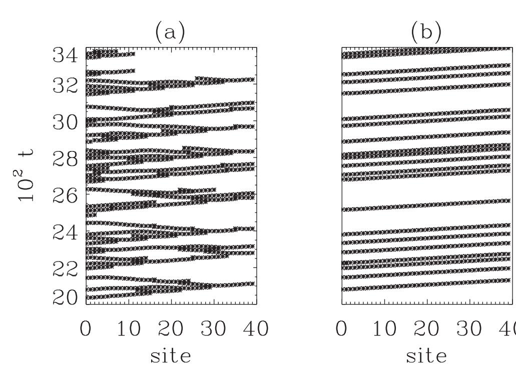

FIG. 1. Space-time position of the spikes for 40 bidirectionally coupled Adler systems with $\mu=0.97$ and the coupling strength (a) $\epsilon=0.75$ and (b) $\epsilon=1.05$.

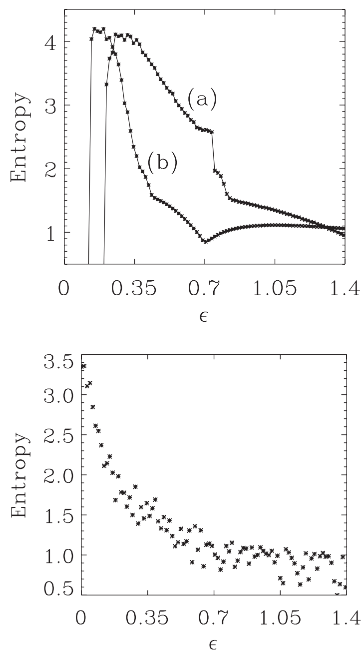

FIG. 2. Top panel: entropy versus coupling strength ϵ for 40 (a) bidirectionally and (b) unidirectionally coupled Adler systems. Bottom panel: entropy versus coupling strength ϵ for the case of 40 coupled Adler systems being in the oscillatory state.

do not mean "isochronous," in which case the space-time plot would be a collection of strictly horizontal lines, but rather that adjacent sites have spikes separated by a fixed lag time.

We observe that the transition to synchronization is not continuous as the control parameter, in our case the coupling strength ϵ, is varying. To describe quantitatively these abrupt changes we characterize the degree of order in the system by means of entropy S. It is calculated from the distribution of the generation times T_g in the time series, far beyond the initial transient. We define the generation time T_g as the time difference between spike occurrences at neighboring sites. When the coupling is zero, this distribution is flat, i.e., the information on a site gives no information on the other ones. Increasing the coupling, we observe the birth of peaks for fixed time differences, due to the time correlation between spikes at adjacent sites. The entropy S is defined as

$$S = -\sum_{T_g} \rho(T_g) \ln \rho(T_g), \tag{2}$$

where $\rho(T_g)$ is the discrete probability distribution (histogram) of a continuous variable T_g. An additional constant contribution, $\ln \Delta T_g$, depending on the discretization step ΔT_g, is not included in the definition of S, since it contributes an irrelevant constant shift. In Fig. 2(a) we plot entropy versus coupling strength in the case of the bidirectionally coupled Adler systems. Above the critical value of the coupling $\epsilon_c = 0.75$ we observe a discontinuous change in the entropy value, thus suggesting the existence of a bifurcation.

For small coupling strength the firing excited at the first site by noise cannot propagate through the array, thus the entropy does not exist (there is no data to analyze). Now, the

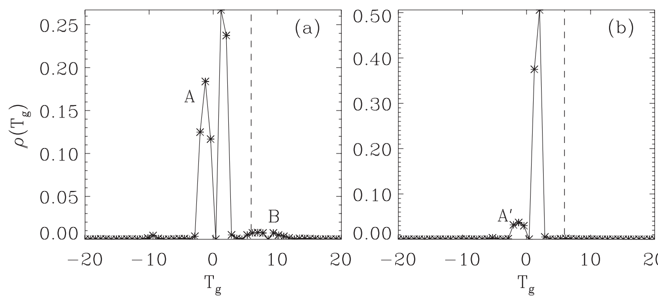

FIG. 3. Coupled Adler systems. Distribution of the generation times T_g for the coupling strengths (a) $\epsilon = 0.75$ and (b) $\epsilon = 0.81$. Dashed lines: refractory period T_r.

interesting observation is made when studying the unidirectional coupling ($n = \{-1\}$). In Fig. 2(b) we observe the monotonic decrease of the entropy as the coupling strength ϵ increases. No sharp transition is observed in this case. This observation suggests that the feedback effects in the case of bidirectional coupling makes synchronization difficult to establish, since additional firings in the two directions may be induced. We notice that these additional firings disappear as soon as the generation time becomes smaller than the refractory period. The refractory period T_r is the time fraction of each cycle during which the system is insensitive to an external perturbation. Thus we state that the above condition is responsible for the occurrence of bifurcation. It is interesting to note that in the case of coupled Adler systems in the oscillatory regime ($|\mu| > 1$) with randomly distributed frequencies, the bifurcation is absent, indeed (Fig. 2, right panel) the transition to synchronization has no sudden change in entropy. The probability distribution in the case of periodic systems is calculated by considering T_g as a phase difference between adjacent oscillators. The lack of bifurcation is due to the fact that the single units of an array do not contain any refractory period.

To prove our statement we plot in Fig. 3 the change of the generation time distributions for different coupling strengths, below and above the critical value ϵ_c. Slightly above ϵ_c, the generation time T_g become smaller than the refractory period T_r (marked by the vertical dashed lines). T_r is defined as the minimum value of time after which the second firing can be generated. In the case of Adler system with $\mu = 0.97$ it is $T_r = 2\pi - 2|\arccos \mu| = 5.8$. In Fig. 3(a), T_g has a nonzero distribution for $T_g > T_r$ (region B). The generation times included in B induce additional firings with negative generation times (peak A). In Fig. 3(b), where ϵ is slightly above ϵ_c, the region B vanishes completely, thus all T_g result to be smaller than T_r. Vanishing of region B induces the sharp decrease of the peak A toward A' [Fig. 3(b)]. This is related to the fact that the feedback is no longer operating and the activation of sites cannot propagate in the opposite direction (defined by negative generation times). The different propagation direction appearing in an array can be read from Fig. 1(a). This observation provides information on the influence of the feedback in the bidirectional coupling. When the generation time T_g on site i is larger than the refractory period on site $i-1$, then the site i can induce successively the second spike on site $i-1$. As the generation time decreases, the excitation of the successive spikes becomes impossible and the effect of the bidirectional coupling vanishes; the array reduces to the unidirectional one. In fact, in the case of unidirectional coupling even for small coupling strengths the excitation from the first site propagates easily only in one direction.

Based on the previous considerations we state that the necessary condition for the bifurcation is that for some small coupling strength some part of the generation time distribution be larger than the refractory period. In order to estimate analytically the condition for the occurrence of a bifurcation we consider the Adler system under the effect of a single perturbation $I(t)=\epsilon\delta(t-t_0)$ acting at time t_0, where the δ function is an idealization of the pulses coming form the neighboring sites. The effect of this perturbation appears only as a discontinuity of the $x(t)$ variable at time t_0 as $x(t_0^+)=x(t_0^-)+\epsilon$. The condition for the perturbation to be larger than the excitability threshold is that $x(t_0^+)>x_+$, where x_+ is the unstable fixed point of Eq. (1). We set the initial condition to be in the rest state, $x(t_0^-)=x_-$, where x_- is the stable fixed point of Eq. (1), such that the minimum value for the amplitude in order to excite a pulse is $\epsilon>2\arccos\mu$ and the system develops a pulse after a certain T_g. This time is defined as the time it takes $x(t)$ to reach a given reference value, e.g., $x_r=\pi/2$. From Eq. (1) for the uncoupled Adler system with $D=0$ we have $T_g=\int_{x(t_0^+)}^{\pi/2}\frac{dx}{\mu-\cos x}$. If for any ϵ the condition

$$T_g=\frac{1}{\sqrt{1-\mu^2}}\ln\left[\frac{(1-b)\left(b^{-1}\tan\frac{x(t_0^+)}{2}+1\right)}{(1+b)\left(b^{-1}\tan\frac{x(t_0^+)}{2}-1\right)}\right]>T_r \quad (3)$$

with $b=\sqrt{\frac{1-\mu}{1+\mu}}$ is satisfied, then the bifurcation will occur as ϵ increases. We also checked the relation between T_g and T_r in FitzHugh-Nagumo system [12] being in an excitable regime. We find, both by analytical calculations and numerical simulations, that in this system $T_g<T_r$ for all system parameters. This fact prohibits the appearance of the bifurcation that we observe in the Adler system.

Furthermore, we show that the bifurcation can appear in coupled chaotic maps with an artificially introduced refractory period. The one-dimensional array of bidirectionally coupled chaotic maps is defined as follows:

$$x_i^{n+1}=a_1x_i^n+a_2(x_i^n)^2+\epsilon(y_{i+1}^n+y_{i-1}^n-2y_i^n),\quad \text{if } x_i^n\leqslant 1,$$

$$x_i^{n+1}=b(x_i^n-1)+c,\quad \text{if } x_i^n>1, \quad (4)$$

where a_1, a_2, b, and c are constant parameters. b is a squeezing factor which reinjects the dynamical point close to the origin, whenever x becomes greater than 1. Variable y takes two values, 1 whenever x crosses 1, and 0 elsewhere; this leads to the coupling scheme of pulse type. In order to account for the refractory region when a spike is generated, the variable x is frozen for T_r steps, then it restarts its cycle.

We evaluate the distributions of the generation times and calculate the corresponding entropy as a function of ϵ for different values of T_r. We find that the slope suddenly decreases above ϵ_c, i.e., the intersite correlations rapidly increase, indicating the beginning of synchronization between the map units. In Fig. 4, the entropy of the time difference distribution is reported for some values of the parameters. For convenience, in Fig. 4 we have vertically translated the curve (b) with respect the other ones to avoid overlaps. When we set the refractory period to zero the bifurcation does not occur at all. It is worth noting that the sharpness and the value of critical coupling strength value of the onset of synchronization may also depend on the grade of fluctuations existing in the system [13].

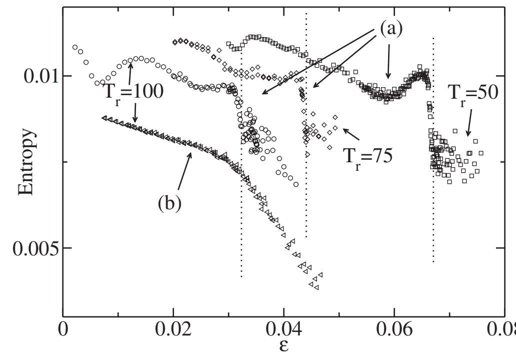

FIG. 4. Coupled chaotic maps with $a_1=1.001$, $a_2=0.3$. Entropy versus ϵ for (a) $b=0.03$ and (b) $b=0.1$. For (a) the three values of T_r, 50, 75, and 100, are considered. For (b) T_r is equal to 100. The vertical lines correspond to the values of ϵ for which $T_r=T_g$.

With the use of a simple excitable system as well as chaotic maps with artificial refractory period we have shown that T_r plays a crucial role in the synchronization process. This provides an explanation for the occurrence of a bifurcation in the bidirectionally coupled systems exhibiting homoclinic chaos (HC). As reported in [9], the sudden transition to synchronization occurs at the critical value of the coupling strength ϵ_c [see Fig. 5(a)]. Following the same considerations as in the case of the Adler system, we notice that again the bifurcation occurs when all T_g become smaller than T_r of a single HC unit. In this case, T_r is defined as the region of low susceptibility to the external stimulus. In Fig. 5(b) we calculate numerically the susceptibility in all regions of the phase space for the HC system (with parameters as in Ref. [9]) and obtain a refractory period of the order of $T_r\approx 30$. The high values of the correlation r correspond to the low susceptibility of the HC system. The correlation r at time t is evaluated between the perturbed time series x_t and the unperturbed one x, both starting from the same initial conditions.

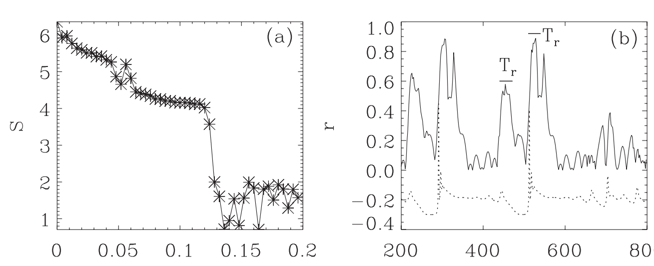

FIG. 5. (a) Entropy S versus coupling strength ϵ for 100 bidirectionally coupled HC systems. (b) Correlation r (solid line) and unperturbed time series (dotted line and out of scale) versus time t.

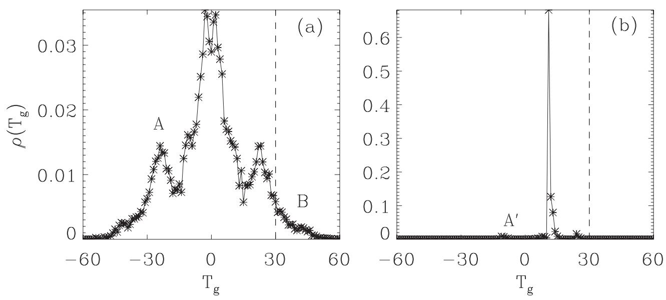

FIG. 6. Distribution of the generation times T_g in 100 HC coupled system for the coupling strengths (a) $\epsilon=0.08$ and (b) $\epsilon=0.14$. Dashed line marks the refractory period.

An external perturbation is applied at time t where r is evaluated. The correlation r is defined as

$$r(t)=\frac{\frac{1}{T}\int_t^{t+T} d\bar{t}x_t(\bar{t})x(\bar{t})-\bar{x}_t\bar{x}}{s_t s}, \qquad (5)$$

where $x_t(\bar{t})$ and $x(\bar{t})$ are dynamical variables, $\bar{x}_t$ and $\bar{x}$ are temporal averages over interval T, and s_t and s are the standard deviations of x_t and x, respectively. T is of order of an average interspike interval of the HC system. In Fig. 6 we plot the distributions of T_g for two coupling strengths, below and above the critical value $\epsilon_c=0.13$. We observe that slightly above ϵ_c the generation times T_g become smaller than T_r. In Fig. 6(a) T_g has a non-zero distribution for $T_g>T_r$ (marked by B). The generation times included in B induce additional firings with negative generation times (marked by A). In Fig. 6(b), where ϵ is set slightly above ϵ_c, the distribution at B vanishes completely, thus all T_g are always smaller than T_r. Vanishing of B induces the sharp decrease of A toward A' [Fig. 6(b)]. As in the Adler system, also in HC the feedback interaction is no longer active and the activation of sites cannot propagate in the opposite direction (defined by negative generation times).

In conclusion, we have explored the role of the interplay between generation times and refractory periods on the cooperation between locally coupled oscillators leading to global synchrony. We have shown that in a one-dimensional array with a finite refractory period, a bifurcation during transition to synchronization occurs at a critical value of the coupling strength when a specific constraint is fulfilled, namely, once the generation times become smaller than the refractory period. On the contrary, globally coupled systems undergo a sudden transition even in the absence of a refractory period, thus for them our criterion is irrelevant. Due to the sparse (even though not just nearest neighbors) coupling of neurons, we show that the criterion here discussed may be an important ingredient to explain the sudden appearance of a coherent perception.

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[1] A. T. Winfree, J. Theor. Biol. **16**, 15 (1967); Y. Kuramoto, Prog. Theor. Phys. Suppl. **79**, 223 (1984).
[2] W. Singer, *Synchrony, oscillations, and relational codes*, edited by L. M. Chalupa and J. S. Werner, The Visual Neurosciences (MIT Press, Cambridge, MA, 2004), pp. 1665–1681.
[3] F. T. Arecchi, Physica A **338**, 218 (2004).
[4] P. Fries *et al.*, Nat. Neurosci. **4**, 194 (2001).
[5] M. V. Ivanchenko, G. V. Osipov, V. D. Shalfeev, and J. Kurths, Phys. Rev. Lett. **93**, 134101 (2004).
[6] T. Shibata, T. Chawanya, and K. Kaneko, Phys. Rev. Lett. **82**, 4424 (1999).
[7] P. Östborn, Phys. Rev. E **66**, 016105 (2002).
[8] A. Montina, C. Mendoza, and F. T. Arecchi, Int. J. Neural Syst. **17**, 79 (2007).
[9] M. Ciszak, A. Montina, and F. T. Arecchi, e-print arXiv:0709.1108v1.
[10] J. F. Mitchell, K. A. Sundberg, and J. H. Reynolds, Neuron **55**, 131 (2007); M. A. Sommer, Neuron **55**, 6 (2007).
[11] M. C. Eguia and G. B. Mindlin, Phys. Rev. E **61**, 6490 (2000).
[12] R. FitzHugh, Biophys. J. **1**, 445 (1961); J. Nagumo, S. Arimoto, and S. Yoshizawa, Proc. Aust. Assoc. Neurol. **50**, 2061 (1962).
[13] I. Z. Kiss, Y. Zhai, and J. L. Hudson, Science **296**, 1676 (2002); A. Serletis, A. Shahmoradi, and D. Serletis, Chaos, Solitons Fractals **33**, 914 (2007).